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Structure and evolution of singular vectors in the Eady model

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SUMMARY

The singular vectors of the propagation matrix of a simple two-dimensional model of the midlatitude atmospheric flow are calculated, in order to obtain initial conditions that achieve the maximum stream-function amplification over a 24 h time period. Simple plane wave initial conditions are found to be a reasonable model for the amplification mechanism of the leading singular vector structures. The primary mechanism for growth is found to be the 'untilting' of the initial potential vorticity wavefronts. The existence of boundary wave structures is found to be the deciding factor in the zonal scale of perturbations and some evidence that these act against the primary growth mechanism is presented.

KEY WORDS: singular vectors; optimal growth structures; midlatitude flow; Eady model

1. INTRODUCTION

In recent years the singular value decomposition of the propagation matrix of the linearized dynamics of fluid flow has found several uses within meteorological circles, e.g. the creation of perturbations for ensemble forecasting [1] or the identification of target regions for adaptive observation strategies [2]. The motivation for the use of this matrix decomposition lies in the fact that right singular vectors form a complete orthogonal set of 'initial' phase space directions, with the first singular vector giving the direction of maximal growth over a finite time interval.

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Due to the fact that the singular vectors are orthogonal, linear combinations of these vectors cannot amplify by a factor greater than the largest singular value associated with the vectors in that combination. Unlike eigenvectors, singular vectors are orthogonal even for the non-symmetric operators associated with most fluid flows. The structure of the singular vectors is dependent on both the time interval and the norm used in their computation.

The dynamical properties of these optimal growth structures are investigated in this paper in the context of a simple two-dimensional model of the midlatitude atmospheric flow. Sections 2 and 3 give brief descriptions of the model formulation and the singular vector computation, respectively. Results and conclusions are contained in Sections 4 and 5.

2. 2D ‘EADY’ MIDLATITUDE FLOW MODEL

The ‘Eady’ model is a simplified quasi-geostrophic representation of the mean atmospheric (incompressible) flow in the northern midlatitude region [3]. In this model the mean temperature field is characterized by a constant negative gradient in the meridional (y) direction and the mean vorticity (Coriolis parameter), f_0, is assumed constant. Through a ‘thermal wind balance’ relationship, a mean wind velocity field in the zonal (x) direction, characterized by a uniform shear with height (z), is induced by the meridional temperature gradient. Additional simplifications can be made by defining the meridional derivative of all perturbations to the background state to be zero, defining the vertical velocity to be zero on the upper and lower boundaries and imposing periodic boundary conditions in the zonal direction. This leads to the linear perturbation evolution equations

$$\left[\frac{\partial}{\partial t} + \bar{u}(z)\,\frac{\partial}{\partial x}\right] q(x,z,t) = 0, \quad z \in (-0.5, 0.5) \tag{1}$$

$$q(x,z,t) = \frac{\partial^2 \psi}{\partial x^2}(x,z,t) + \frac{\partial^2 \psi}{\partial z^2}(x,z,t), \quad z \in (-0.5, 0.5) \tag{2}$$

$$\left[\frac{\partial}{\partial t} + \bar{u}(z)\frac{\partial}{\partial x}\right] \frac{\partial \psi}{\partial z}(x,z,t) - \frac{\partial \psi}{\partial x}(x,z,t) = 0, \quad z = \pm 0.5 \tag{3}$$

for $x \in [0, X]$, where $\bar{u}(z) = z$ is the mean zonal wind, $q(x,z,t)$ is a quasi-geostrophic potential vorticity perturbation, and $\psi(x,z,t)$ is a stream-function perturbation. The partial stream-function derivative with respect to x is the meridional velocity perturbation, $v(x,z,t) = \psi_x(x,z,t)$, and that with respect to z is the buoyancy (thermal) perturbation, $b(x,z,t) = \psi_z(x,z,t)$. The zonal co-ordinate is non-dimensionalized by a factor NH/f_0, the vertical co-ordinate by H and time by $N/f_0\Lambda$, where $N = 10^{-2}\mathrm{s}^{-1}$ is a vertical stability parameter, $H = 10^4$ m is the height of the domain, $f_0 = 10^{-4}\,\mathrm{s}^{-1}$ is the mean vorticity and $\Lambda = 4 \times 10^{-3}\mathrm{s}^{-1}$ is the vertical shear of the mean wind field. The vertical co-ordinate is shifted so that zero is in the centre of the domain. The inversion of Equation (2) subject to periodic and derivative boundary conditions is made unique by enforcing the condition that the stream-function field has a zero mean value. For the numerical results presented in this paper a discrete grid-point model, with 50 grid-points horizontally and 41 grid-points vertically on a domain of non-dimensional horizontal extent $X = 6$ corresponding to a dimensionalized distance of 6000 km was used.

3. SINGULAR VECTOR COMPUTATION

The state at time t of a discretization of the model is defined by a vector, $\boldsymbol{\psi}(t) \in \mathbb{R}^n$, of grid-point stream-function values. An initial state at time $t=0$ is propagated forward in time to the state at time $t=\tau$, by multiplication by the matrix $L_{0,\tau} \in \mathbb{R}^{n\times n}$. By performing a singular value decomposition [4] on this matrix

$$L_{0,\tau} = U\Sigma V^{\mathrm{T}} \tag{4}$$

a complete set of orthonormal vectors, $\boldsymbol{v}_i$, ('right singular vectors'; columns of $V \in \mathbb{R}^{n\times n}$) representing components of the state at $t=0$, are obtained. These evolve to a corresponding set of orthonormal vectors, $\boldsymbol{u}_i$, ('left singular vectors'; columns of $U \in \mathbb{R}^{n\times n}$) with 2-norm amplification of each vector given by the singular values, σ_i (the diagonal entries of $\Sigma \in \mathbb{R}^{n\times n}$).

Since the right singular vectors form a complete basis for $\mathbb{R}^n$, all initial states may be expressed as a linear combination of these vectors. Hence the initial and corresponding final state may be written as

$$\boldsymbol{\psi}(0) = \sum_{i=1}^{n} c_i \boldsymbol{v}_i \quad \text{and} \quad \boldsymbol{\psi}(\tau) = \sum_{i=1}^{n} c_i \sigma_i \boldsymbol{u}_i \tag{5}$$

respectively, where $c_i = \boldsymbol{v}_i^{\mathrm{T}} \boldsymbol{\psi}(0)$. Due to the orthogonality of the singular vectors the amplification of this mapping, as measured in the vector 2-norm, may be expressed as

$$\frac{\|\boldsymbol{\psi}(\tau)\|_2}{\|\boldsymbol{\psi}(0)\|_2} = \left(\sum_{i=1}^{n} c_i^2 \sigma_i^2 \Big/ \sum_{i=1}^{n} c_i^2 \right)^{1/2} \tag{6}$$

By convention the singular values are indexed such that $\sigma_i \geqslant \sigma_{i+1}$. Therefore, simple arguments can be used to show that the first right singular vector, $\boldsymbol{v}_1$, is the initial state that achieves the maximum amplification over the time interval $[0, \tau]$.

4. RESULTS AND ANALYSIS

A complete set of singular vectors were calculated for a time interval of twenty-four hours. The potential vorticity (pv) and stream-function evolution associated with the first singular vector are shown in Figures 1(a) and (b), respectively. This evolution may be summarized as follows: the initial stream-function field is associated with the interior potential vorticity structure and not with thermal waves on the upper and lower boundaries; this pv structure, initially tilted against the mean wind shear, is rotated to a vertical position over the twenty-four hour period, via the conservation law (1); this is associated with a rotation and amplification of the initial stream-function field; by twenty-four hours, stream-function structures on the upper and lower boundaries associated with thermal waves are clearly visible.

An understanding of the growth mechanism of the leading singular vectors can be gained by comparison with the evolution of similar plane wave pv structures, which have a known continuous functional form. The stream-function field associated with such structures [5] is given by

$$\psi(x,z,t) = \left\{ \frac{(1+a_0^2)}{[1+(a_0-t)^2]} \mathrm{e}^{\mathrm{i}k(a_0-t)z} + A(t)\cosh(kz) + B(t)\sinh(kz) \right\} \mathrm{e}^{\mathrm{i}kx} \tag{7}$$

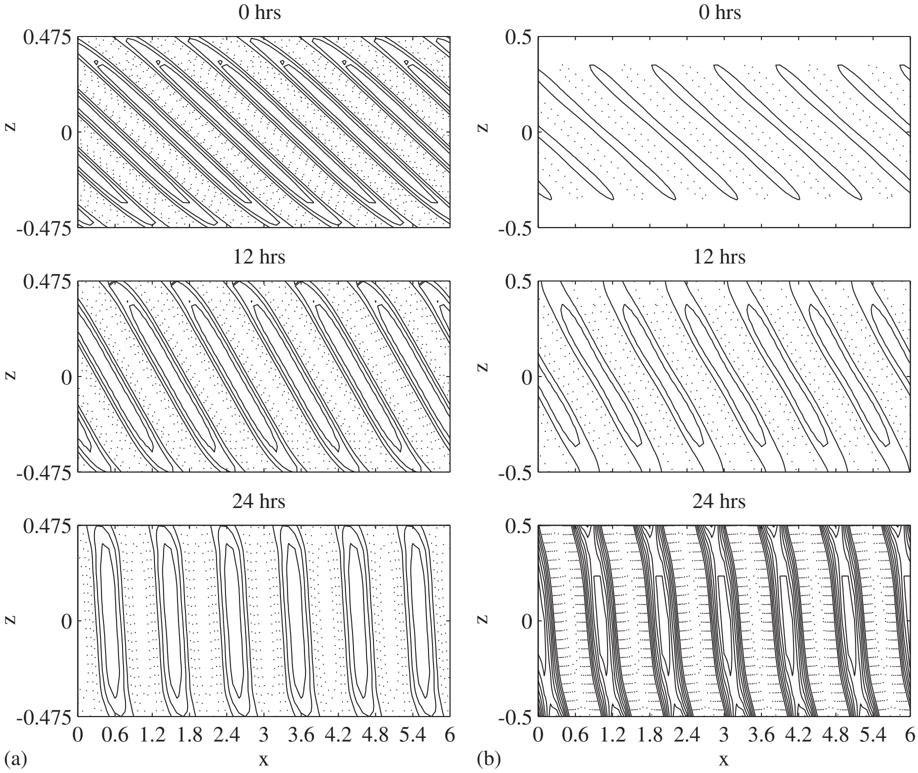

Figure 1. The potential vorticity (a) and stream-function (b) evolution of the 1st singular vector. Solid contours denote positive values, dotted contours denote negative.

where k is the zonal wavenumber, m_0 is the initial vertical wavenumber and $a_0 = m_0/k$ is the ratio of the two. The first term in the bracket in Equation (7) gives the structure of the stream-function associated with the interior pv. The other two terms give the stream-function structure due solely to pv independent thermal perturbations on the upper and lower boundary.

For reference Figure 2(b) shows an initial stream-function structure of the form of Equation (7), with the values $a_0 = 3.25$, $k = 2\pi$, $A(0) = B(0) = 0$ chosen specifically to match as closely as possible the first singular vector structure. The evolved stream-function field at $t = 24$h is also shown. As with the singular vector, the stream-function wave fronts are rotated to a nearly vertical position by $t = 24$ h. The thermal structures on the boundary at $t = 24$ h are also present and are more pronounced than in the singular vector.

The evolution of the stream-function associated with the pv for this structure is readily interpretable by examination of Equation (2). The initially tilted plane waves are associated with high ψ_{zz}. As the plane waves are untilted and reach vertical position, $\psi_{zz} \to 0$. Therefore, because pv is conserved, ψ_{xx} must increase, which implies that, since the horizontal structure is sinusoidal, the amplitude of ψ itself must be increasing to compensate for the reduction in vertical gradients.

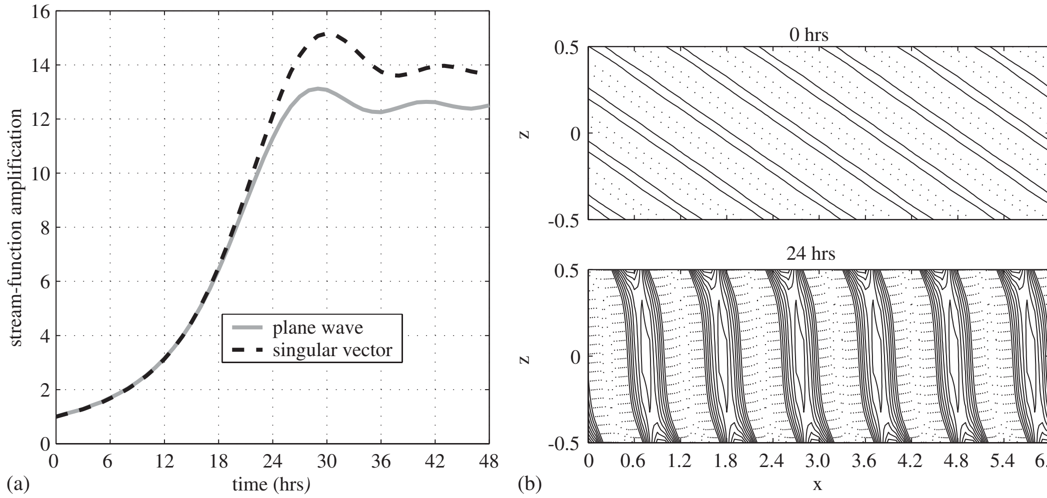

Figure 2. (a) Plane-wave and first singular vector stream-function amplification; and (b) plane-wave stream-function structure at $t = 0$, $t = 24$ h.

A result of this stream-function amplification is an increase in the forcing at the boundaries due to the meridional velocity, implying that although initially zero, $A(t)$ and $B(t)$ do not remain so [5]. This means that as the vertical stream-function gradient associated with the pv tends to zero, the boundary waves associated with the second two terms of Equation (7) become increasingly prominent within the overall stream-function structure. As the pv crosses the vertical position and begins to tilt 'with' the wind shear beyond 24 h (not shown), the transfer from ψ_{zz} to ψ_{xx} happens in reverse, leading to reduction in amplitude of the stream-function associated with the pv, so that only the (in this case) neutral boundary waves remain.

Figure 2(a) shows the evolution of stream-function 2-norm amplitude against time for both the first singular vector and the plane wave structure. There is little distinction between the two curves until ~20 h, implying that it is not until the wavefronts have reached almost vertical orientation, that the difference in structure between the two initial conditions becomes significant. The explanation for this eventual discrepancy between the evolution of the two structures must lie in the fact that the singular vector has a reduced stream-function amplitude towards the upper and lower boundaries, compared to the plane-wave, since this is the only significant difference. This reduces the amplitude of the ψ_x forcing to Equation (3) leading to a reduction in the amplification of the thermal waves on the boundaries. The likely effect of this is to delay the inevitable domination of the overall structure by the neutral boundary waves leading to a slight increase in the final stream-function amplitude. Figure 2(a) seems to confirm this, in that the singular vector stabilizes several hours after the plane wave, and at a higher amplitude.

Figure 3(b) shows the leading eight singular values (stream-function amplification) ordered by the number of zonal wavelengths, ι, in the corresponding singular vector; here $k = 2\iota\pi/6$. There is a noticeable variation in the amplification achieved by perturbations of different zonal scales. Comparison with the plane-wave initial conditions can give some indication as to the cause of this scale dependence. Figure 3(a) shows the 24 h amplification of initial

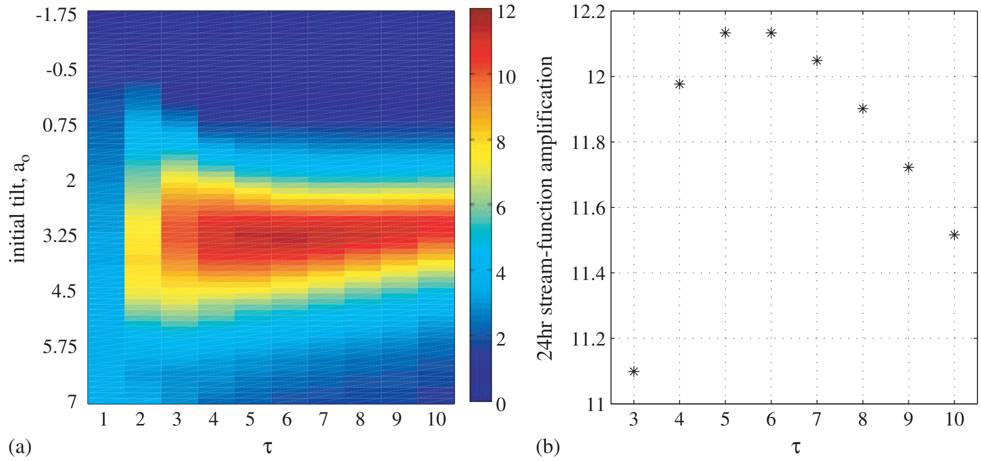

Figure 3. (a) Plane-wave stream-function amplification against a_0 and $\imath = 6k/2\pi$; and (b) leading singular vector stream-function amplification against $\imath = 6k/2\pi$.

perturbations of the form of Equation (7), with $A(0)=B(0)=0$, as a function of both $\imath$ and initial tilt a_0. There is a strong similarity in the amplification scale dependence of the singular vectors and that of plane-waves with initial tilts $a_0 \sim 3.25$, suggesting that the former may be well modelled by the latter. Inspection of Equation (7) indicates that this scale dependence is attributable to the growth of boundary waves, since, unlike the pv associated with the stream-function, the amplitude of the boundary waves varies with wavenumber, k.

The evolution of waves on the upper and lower boundaries, within this model has a marked scale dependence. Long waves (in this particular model setup $\imath = 1,2$) exhibit exponential growth/decay, whereas short waves ($\imath > 2$) are neutrally stable and amplify only as a result of the meridional velocity associated with the interior pv. It is noticeable that the leading singular vectors do not have zonal scale large enough to excite exponentially growing boundary waves. This, and the significant reduction in plane-wave amplification associated with $\imath = 1,2$, suggest that the development of boundary waves acts to reduce the effect of the primary 'pv untilting' growth mechanism.

5. CONCLUSIONS

The 24 h optimal growth perturbations for the 'Eady' midlatitude flow model, defined by the singular vectors of the propagation matrix, are well modelled by simple plane-wave initial conditions. There is a direct correspondence between initial tilt and zonal scale of the first singular vector and that of the 24 h optimal growth plane-wave initial condition. The growth achievable by initially tilted plane-wave like structures of differing zonal scales, seems to be determined by the thermal waves that develop on the boundary. These thermal waves appear to act against the primary untilting growth mechanism, which is a likely explanation for the reduction in the stream-function amplitude (and hence thermal wave growth) towards the upper and lower boundaries.

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